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Optimizing Feedback Linearization for MIMO Nonlinear Tracking Control Using Input-Output and Input-State Techniques

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Abstract

Feedback linearization is a valuable method utilized in control systems to convert the dynamics of nonlinear systems into a linear format, simplifying their analysis and control. However, managing highly nonlinear systems can be both challenging and complex. This paper seeks to address this challenge by proposing an enhanced approach to the feedback linearization technique. To improve the feedback linearization tracking control of multi-input multi-output (MIMO) nonlinear systems, the paper investigates two primary strategies. The first strategy involves adjusting control gains mathematically along with other parameters to optimize control performance, enabling precise tuning of the system's behavior and response to achieve specific objectives. The second strategy focuses on evaluating the performance of feedback linearization control through simulations across various scenarios, disturbances, and reference inputs. By conducting these simulations, researchers can thoroughly examine how the system behaves and performs under different conditions. It is essential to maintain system stability throughout these adjustments and simulations. The paper examines two feedback linearization techniques: input-output linearization and input-state linearization. Each method provides distinct advantages and trade-offs depending on the system's characteristics and control objectives. By applying these techniques, the goal is to achieve the desired tracking performance and behavior of a

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nonlinear system with three inputs and three outputs, which serves as the primary application in this work.

Keywords: Linear Systems, Nonlinear Systems, Tracking Control, Feedback MIMO Systems, Stability Concept, Environmental Disturbances, Control Optimization.

تحسين التغذية العكسية الخطية لتتبع نظم التحكم الغير خطية ذات المداخل والمخارج المتعددة بأستخدام تقنيات الدخل والخرج وحالة الدخل

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الملخص

تعتبر إعادة التشكيل الارتجاعي هي طريقة قيّمة تُستخدم في أنظمة التحكم لتحويل ديناميكيات الأنظمة غير الخطية إلى صيغة خطية، مما يُبسّط تحليلها والتحكم فيها. ومع ذلك، فإن التعامل مع الأنظمة غير الخطية قد يكون صعبًا ومعقدًا بشكل كبير. يهدف هذا البحث إلى معالجة هذا التحدي من خلال اقتراح نهج محسّن لتقنية إعادة التشكيل الارتجاعي.

ولتحسين التحكم باستخدام إعادة التشكيل الارتجاعي في الأنظمة غير الخطية ذات المداخل والمخارج المتعددة(MIMO) ، يستعرض البحث طريقتين رئيسيتين. تتمثل الطريقة الأولى في تعديل متغيرات التحكم إلى جانب بعض المتغيرات الأخرى لتحسين أداء النظام، مما يتيح ضبطًا دقيقًا لسلوك النظام واستجابته لتحقيق أهداف محددة. أما الطريقة الثانية، فتركز على تقييم أداء التحكم باستخدام إعادة التشكيل الارتجاعي من خلال عمليات محاكاة ضمن سيناريوهات متعددة تشمل بعض المؤثرات الخارجية والمدخلات المرجعية المختلفة. ومن خلال هذه المحاكاة، يمكن للباحثين فحص أداء النظام وسلوكه تحت ظروف تشغيل مختلفة. ومن الضروري الحفاظ على استقرار النظام أثناء هذه التعديلات والمحاكاة. ويناقش هذا البحث تقنيتين محددتين لتصميم التحكم باستخدام إعادة التشكيل الارتجاعي، وهما: إعادة التشكيل الارتجاعي للمدخل–المخرج (Input–Output Linearization) وإعادة التشكيل الارتجاعي للمدخل–الحالة (Input–State

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Linearization)، وتوفر كلاً من هاتين التقنيتين مزايا وتحديات مختلفة وفقًا لمتطلبات وخصائص النظام. ومن خلال استخدام هذه التقنيات، يسعى المصممون إلى تحقيق السلوك والأداء المطلوبين للنظام غير الخطي ذي المداخل والمخارج المتعددة، كما يوفر هذا النظام أداء متميز للتتبع بين المخرجات والمدخلات.

الكلمات المفتاحية: الأنظمة الخطية، الأنظمة غير الخطية، أنظمة التحكم بالتتبع، أنظمة MIMO ذات التغذية العكسية، مفهوم الاستقرار، المؤثرات الخارجية، تحسين أنظمة التحكم

Introduction

In the feedback linearization approach, the primary goal is to stabilize and manage nonlinear systems by converting their dynamics into a linear structure.

This work emphasizes local feedback linearization, where coordinate transformations and control laws are defined only within a localized region to avoid complexities associated with global solutions [6]. Feedback linearization generally relies on two main approaches: input-output linearization and input-state linearization. In input-output linearization, a linear relationship is established between the transformed inputs (v) and the actual outputs (y), followed by designing a linear controller for the resulting linearized input-output model. However, this method often leaves behind a residual nonlinear subsystem that remains unaddressed [3,10]. On the other hand, input-state linearization seeks to linearize the relationship between the transformed inputs and the entire transformed state vector [19]. This is achieved by defining artificial outputs (Y) to create a feedback-linearized model with state dimension $r = n$. Controllers designed via this method are more complex, as the mapping between transformed inputs and original outputs (y) remains nonlinear. Consequently, input-state linearization is less practical compared to input-output methods [8].
The structure of the remaining sections of this paper is as follows:
Section II; presents previous works. Section III presents a detailed discussion on the input-output feedback linearization technique for MIMO systems, providing a thorough understanding of its principles and highlighting its importance in effectively handling nonlinear systems. Section IV discusses the input-state linearization technique for MIMO nonlinear systems. Section V defines the problem statement, focusing on specific MIMO nonlinear systems

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consisting of 3 inputs and 3 outputs. Section VI outlines the simulation process using MATLAB, applying both principal input-output and input-state feedback linearization techniques. It explains the methodology used, the parameters considered, and presents a detailed analysis of the obtained results, including system responses. This section critically evaluates the effectiveness and suitability of the proposed techniques. Lastly, Section VII provides the conclusion, summarizing key findings, discussing their implications, and highlighting the study's contributions. Additionally, it suggests potential future research directions and areas for further improvement, serving as a comprehensive conclusion to the paper.

Literature Review

Several methods have been developed to simplify the control of nonlinear systems by transforming them into linear equivalents. Among these, feedback linearization has emerged as a widely studied and applied technique. These techniques allow for the applications of linear control strategies to nonlinear systems as noted in [1, 4]. The method seeks to remove nonlinearities from the system dynamics by identifying an appropriate transformation of variables, as discussed in [1, 2, 3].

As stated in [3], nonlinear MIMO systems can be expressed in the following Equation (1) within continuous-time state-space models.

$$\left.\begin{aligned} \dot{x} &= f(x) + \textstyle\sum_{i=1}^{m} g_i(x)\, u_i, \\ y_i &= h_i(x).\ i = 1,2, \dots, m \end{aligned}\right\} \qquad (1)$$

where: x represents an n-dimensional vector of state variables; u denotes an m-dimensional vector of control inputs or manipulated variables; y corresponds to an m-dimensional vector of output variables; $f(x)$ is an n-dimensional vector of nonlinear functions; $g(x)$ is an $(n \times m)$ dimensional matrix of nonlinear functions; and $h(x)$ is an m-dimensional vector of nonlinear functions.

For MIMO, m in Equation (1) is the number of inputs and outputs; on the other hand, feedback linearization offers the advantage of generating a linear model that accurately mirrors the original nonlinear system across a wide range of operating conditions. The method involves two sequential steps: the first step focuses on

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modifying the system's nonlinear coordinates, and the second step, the nonlinear state feedback, is implemented [4, 5].
Various techniques have been implemented in this field, producing a range of outcomes. For instance, the authors in [1, 4] provide a detailed explanation of the feedback linearizing control principle. In [2], Delgado, and others present input-output linearization of MIMO systems with Applications to longitudinal flight dynamics. The nonlinear control of MIMO systems using feedback linearization and a PD controller for tracking is introduced by Ghozlane, Wafa, and Jilani Knani in [3]. In [5], Horacio J. and Marquez explore the analysis and design of nonlinear control systems. Meanwhile, Wang, Jianliang, and W. Hassan Khalil discusses adaptive output feedback control for nonlinear systems in [6, 7]. Rugh and Jianliang present feedback linearization approaches for nonlinear systems in [8]. Also, Sastry and Shankar analyze the stability and control of nonlinear systems in [9].
Additional details on feedback linearization for nonlinear MIMO variables are available in [10], [11], and [12]. Furthermore, adaptive MIMO nonlinear systems utilizing fuzzy logic control and extreme learning machines are discussed in [14, 15]. Lastly, output feedback linearization for neural network-based ANARX models and nonlinear control for output voltage regulation of a boost converter with a constant power load are presented in [17, 18], respectively.

Input-Output Feedback Linearization for MIMO Systems (I/O)

In this section, the focus is on the concept of linearization of MIMO nonlinear systems. The primary aim of feedback linearization is to establish a linear relationship among the outputs Y_i and the redesigned inputs V_i as illustrated in Figure 1. For MIMO systems described by Equation 1, f; g_i and h_i are sufficiently smooth within a domain $U < R^n$ [1, 2, 3]. The mappings $f : U \to R^n$ and $g : U \to R^n$ referred to as vector fields on U [13, 16].
By referring to Equation 1 and computing the first derivative of the outputs y_i with respect to x, further analysis can be conducted as follows:

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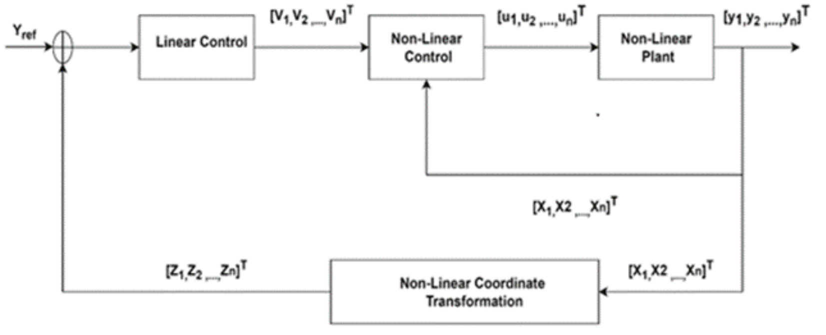

Figure 1. Input-Output Feedback of MIMO Linearization Technique [3].

$$\dot{y}_i = \frac{\partial h_i}{\partial x}\dot{x} = \frac{\partial h_i}{\partial x}\left[f(x) + \sum_{i=1}^{m} g_i(x)u_i\right] \triangleq D_f h_i(x) + \sum_{i=1}^{m} D_g h_i(x)u_i, \quad (2)$$

where:

$$\left.\begin{aligned} D_f h_i(x) &= \frac{\partial h_i}{\partial x}[f(x)], \\ D_g h_i(x) &= \frac{\partial h_i}{\partial x}[g(x)] \\ D_g D_f h_i(x) &= \frac{\partial D_f h_i}{\partial x} g(x) \end{aligned}\right\} \quad (3)$$

Then:

$$\left.\begin{aligned} D_f^0 h_i(x) &= h_i(x), \\ D_f^2 h_i(x) &= D_f D_f h_i(x) = \frac{\partial(D_f h_i)}{\partial x} f(x), \\ &\vdots \\ D_f^k h_i(x) &= D_f D_f^{k-1} h_i(x) = \frac{\partial\left(D_f^{k-1} h_i\right)}{\partial x} f(x). \end{aligned}\right\} \quad (4)$$

Where:

$$\left.\begin{aligned} y_i &= [y_1, y_2, \ldots\ldots, y_p]^T, \\ u_i &= [u_1, u_2, \ldots\ldots, u_p]^T.\ i = 1,2,\ldots,m \end{aligned}\right\} \quad (5)$$

Note: In this work, the number of inputs is equal to the number of outputs.

Definition 1: In Equation 4, if $D_g h_i(x)u_i = 0$, *then* $\dot{y}_i = D_f h_i(x)$ (which is independent of u_i). For more explanation, it is necessary to calculate the second derivatives and higher-order derivatives [1,3].

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$$\left.\begin{aligned} y_i^{(2)} &= \frac{\partial D_f h_i}{\partial x}[f(x) + g_i(x)u_i] \\ &= D_f^2 h_i(x) + D_g D_f h_i(x) u_i. \end{aligned}\right\} \quad (6)$$

Once more, if $D_g D_f h_i(x)\, u_i = 0$, then $y_i^{(2)} = D_f^2\, h_i(x)$ (which is independent of u_i).

Definition 2: If we perform the same calculations for higher derivatives, we obtain the following.

$if\ \ D_g D_f^{i-1} h_i(x) = 0,\ i = 1, 2, \dots, r-1,\ D_g D_f^{r-1} \neq 0,$ then u_i does not appear in $y_i, \dot{y}_i \dots \dots \dots, y_i^{r-1}$.

Equation 6 is modified as:

$$\boldsymbol{y_i^{r-1}} = D_f^r h_i(x) + D_g D_f^{r-1} h_i(x) u_i \quad (7)$$

finally, the control laws can be formed as:

$$u_i = \frac{1}{D_g D_f^{r-1} h_i(x)}\left[-D_f^r h_i(x) + v_i\right]. \quad (8)$$

Then by substituting with the value of u_i that represented in Equation 8, into the nonlinear system that represented in Equation 1, the MIMO systems become input-output linearizable, reducing to $y^r = v$ which forms a chain of r integrator. After the feedback linearization process, the system models become linear in the form:

$$\left.\begin{aligned} \dot{\zeta} &= A\zeta + Bv \\ Y &= C\zeta \end{aligned}\right\} \quad (9)$$

Where: ζ is r- dimensional vector of transformed state variables; v and Y are m- dimensional vectors of transformed input and output variables respectively; and the matrices A, B and C are in simple structures, and for MIMO system can be written as stated in [3].

$$A = \begin{bmatrix} A_{j1} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & A_{jp} \end{bmatrix}, B = \begin{bmatrix} B_{j1} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & B_{jp} \end{bmatrix}, C = \begin{bmatrix} C_{j1} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & C_{jp} \end{bmatrix}$$

And

$$A_{j_i} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{j_i \times j_i},\ B_{j_i} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^{j_i},\ C_{j_i} = [1 \quad 0 \dots \quad 0] \in \mathbb{R}^{j_i}$$

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Stability Analysis:

Stability analysis plays a crucial role in the input-output feedback linearization method. While stability is not the main goal of this technique, it is imperative to guarantee the stability of the closed-loop system once the linearized control design is implemented [9]. This viewpoint is supported by references [4, 5] and [6].

Lemma 1: The nonlinear system defined in Equation 1 is said to have a relative degree $m, 1 \leq m \leq n$ $in\ region\ U_0 < U$ if

$$D_g D^{i-1} h_i(x) = 0, \quad i = 1, 2, \ldots\ldots, m-1.$$

$$D_g D^{m-1} h_i(x) \neq 0 \quad for\ all\ x \in U_0 \ .$$

Lemma 2: The nonlinear system defined in Equation 1, has relative degree $m \leq n$ in the region U. If $m = n$ then for every $x_0 \in U$, a neighborhood *N* of x_0 exists such that:

$$T(x) = \begin{bmatrix} h_i(x) \\ D_f h_i(x) \\ \vdots \\ D_f^{n-1} h_i(x) \end{bmatrix}. \tag{10}$$

Bounded to *N*, is a diffeomorphism on *N* [8, 9].

In addition, *if* $m < n$, then for each $x_0 \in U$, a neighborhood *N* of x_0 and smooth functions.

$\psi_1(x), \ldots \ldots.., \psi_{n-m}(x)$ exist such that $\frac{\partial \psi_i}{\partial x} g(x) = 0, for\ 1 \leq i \leq (n-m)$ for all $x \in N$, and the matrix.

$$T(x) = \begin{bmatrix} \psi_1(x) \\ \vdots \\ \cdot \\ \psi_{n-m}(x) \\ \ldots\ldots\ldots. \\ h_i(x) \\ D_f h_i(x) \\ \cdot \\ \vdots \\ D_f^{m-1} h_i(x) \end{bmatrix} = \begin{bmatrix} \psi(x) \\ \ldots\ldots.. \\ \varphi(x) \end{bmatrix} = \begin{bmatrix} \zeta \\ \ldots\ldots.. \\ \xi \end{bmatrix} \tag{11}$$

where: $\varphi(x) = \begin{bmatrix} h_i(x) \\ D_f h_i(x) \\ \vdots \\ D_f^{m-1} h_i(x) \end{bmatrix} \in \mathbb{R}^{mi}$

Equation 11 is bounded to *N* which is a diffeomorphism on *N*.

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The next step is by taking the derivative for both ζ and ξ variables, we obtain:

$$\left.\begin{aligned} \dot{\zeta} &= f_0(\zeta,\xi) \\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u-\alpha(x)] \\ y &= C_c\xi \end{aligned}\right\} \tag{12}$$

Where; $\xi \in R_m$, $\zeta \in R^{n-m}$, and (A_C, B_C, C_C) represents the controller canonical form of a chain of m integrators.

$$f_0(\zeta,\xi) = \frac{\partial\psi}{\partial x} f(x)|_{x=T^{-1}}(z) \ . \tag{13}$$

Where

$$\gamma(x) = \begin{bmatrix} D_g D_f^{m_{i-1}} h_1(x) \\ \vdots \\ D_g D_f^{m_{ny-1}} h_{ny}(x) \end{bmatrix} \in \mathbb{R}^{n_y \times n_u}, \tag{14}$$

and

$$\alpha(x) = \beta(x) \begin{bmatrix} -D_f^{m1} h_1(x) \\ \vdots \\ -D_f^{mn_y} h_{n_y}(x) \end{bmatrix} \in \mathbb{R}^{n_u}. \tag{15}$$

Known that $\beta(x) \in \mathbb{R}^{n_u \times n_y}$ is the pseudo-inverse of (x) . Finally, the controller u can be written in the form

$$u = \alpha(x) + \beta(x)v. \tag{16}$$

Yields:

$$\dot{\xi} = A_c\xi + B_c\Lambda(x)\, v \tag{17}$$

where $\Lambda(x) \triangleq \gamma(x)\,\beta(x) \in \mathbb{R}^{n_y \times n_y}$ and $v \in \mathbb{R}^{n_y}$. Note that $\Lambda(x)$ can be written as:

$$\Lambda(x) = \begin{bmatrix} \lambda_1(x) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{n_y}(x) \end{bmatrix}, \tag{18}$$

Where, for all $i = 1, \dots, n_y, \lambda_i(x)$ is either 1 or 0.

Referring to the system of equations represented in (11), (12), and (17), the MIMO system is asymptotically stable. Furthermore, for tracking purposes, vector v in Equation 16 must satisfy the following [9, 10]:

$$v_i = y_{m_i}^{n_i} + K_{n_{i-1}}(y_{m_i}^{n_{i-1}} - y_i^{n_{i-1}} + \dots + K_1(y_{m_i} - y_i), \tag{19}$$

with $1 \le i \le p$.

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In addition, $y_{m_i}, y_{m_i}^2, \ldots, y_{m_i}^{n_i}$ are known as different trajectories for different outputs of the system, and the polynomials K_i are chosen to have roots with negative real parts [2]. Thence, the error $e_i(t) = y_{m_i}(t) - y_i(t)$, must satisfied $\lim_{t \to \infty} e_i(t) = 0$, this implies the MIMO system is stable.

Input-State Linearization (I/S)

Input-state linearization is a powerful technique for controlling MIMO nonlinear systems [1,3]. The goal is to transform the nonlinear system into a fully linear system in the state-space domain using a nonlinear coordinate transformation and state feedback, more information about the structure of this technique can be found in [1,2].

In this section, Equation 1 can be reconsidered for the input-state linearizing method as:

$$\left.\begin{aligned} \dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)\, u_i, \\ y_i &= h_i(x).\ i = 1,2,\ldots,m \end{aligned}\right\} \quad (20)$$

Where: $x \in \mathbb{R}^n$, is the state vector, $u \in \mathbb{R}^m$ is the control input vector. $f(x) \in \mathbb{R}^n$ is the drift vector field, $g_i(x)$ are smooth fields, $u = [u_1, \ \ldots, \ u_m]^T$ is the control input and $h(x) \in \mathbb{R}^m$ is the output matrix.

For a successful linearization procedure, two assumptions should be considered in this process.

Assumption 1: (Controllability Condition) To make the nonlinear system fully linearizable using the input-state method, it should be controllable in the nonlinear sense.

Assumption 2: (Involutivity condition) The distribution spanned by the columns of $g(x)$ must be involutive (i.e. the lie bracket any two columns of $g(x)$ must lie in the span of $g(x)$ [4].

To linearize the nonlinear system in Equation 20, we define the following transformation.

$$z = T(x), \quad (21)$$

Where: $z \in \mathbb{R}^n$, is the new state vector in the transformed coordinates.

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$$T(x) = [T_1(x) \quad T_2(x) \ldots \quad T_n(x)]^T \tag{22}$$

And

$$Z_j^i = D_f^{j-1} T_i(x), \tag{23}$$

where $i = 1, \ldots, m$; $j = 1, \ldots, r_i$.

The system defined in Equation 22 and Equation 23 is a linear system, and the next step is to compute the Lie derivative as follows:

$$\left.\begin{aligned} \dot{Z}_i &= Z_{i+1}, \\ \dot{Z}_{i+1} &= Z_{i+2}, \\ &\cdot \\ &\vdots \\ \dot{Z}_n &= v_i \,. \end{aligned}\right\} \tag{24}$$

Where: $v_i = [v_1, \quad \ldots, \quad v_m]^T$, is the new control inputs in the transformed coordinates. Equation 24 can be rewritten in the form:

$$\dot{Z} = A\,z + B\,v. \tag{25}$$

Where, $A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}$.

Then the new control input *v* can be formed as:

$$v = -KZ, \tag{26}$$

where *K* is the feedback gain matrix, and the control law in Equation 16 is recalled and modified as:

$$u = G^{-1}(x)(-b(x) + v). \tag{27}$$

where $G(x)$ is the decoupling matrix, which is defined as:

$$G_{ij}(x) = D_{gi}\, D_f^{r_i - 1} T_i(x), \tag{28}$$

where $G(x) \in \mathbb{R}^{m \times m}$, and $b(x)$ is the nonlinearity vector which denoted as:

$$b_i(x) = D_f^{r_i} T_i(x), \tag{29}$$

where $b(x) \in \mathbb{R}^m$; finally, if we compare Equation 16 with Equation 27, we obtain the following:

$$\alpha(x) = -G^{-1}(x) b(x) \quad and \quad \beta(x) = G^{-1}(x).$$

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Problem Statement

In this section, we introduce the problem statement by examining a representative example involving MIMO nonlinear system and analyzing the resulting outcomes. To begin, let us consider the following nonlinear MIMO system:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + u_3 \\ \dot{x}_2 &= -x_1x_3 + x_2^2 + u_1 \\ \dot{x}_3 &= x_1x_2 + u_2 \\ y_1 &= x_2 \\ y_2 &= x_3 \\ y_3 &= x_1x_2 \end{aligned}\right\}, \qquad (30)$$

Where x_1 , x_2 and x_3 are the system states, u_1, u_1 and u_3 are the system controls, and y_1, y_1 and y_3 are the system outputs.

Then, the tracking errors can be defined as:

$$\left.\begin{aligned} e_1 &= y_{1-d} - x_1 \\ e_2 &= y_{2-d} - x_2 \\ e_3 &= y_{3-d} - x_3, \end{aligned}\right\} \qquad (31)$$

Where y_{1-d}, y_{2-d} and y_{3-d} are the system reference trajectories. By combining Equations 30 and 31, we obtain the following:

$$\left.\begin{aligned} \dot{x}_1 &= \dot{y}_{1-d} + k_1(y_{1-d} - x_1) \\ \dot{x}_2 &= \dot{y}_{2-d} + k_2(y_{2-d} - x_2) \\ \dot{x}_3 &= \dot{y}_{3-d} + k_3(y_{3-d} - x_3) \end{aligned}\right\}, \qquad (32)$$

Where k_1 , k_2 and k_3 are the feedback system gains. The next step is to calculate the error dynamics, which can be stated as:

$$\left.\begin{aligned} \dot{e}_1 &= \dot{y}_{1-d} - \dot{x}_1 = -k_1e_1 \\ \dot{e}_2 &= \dot{y}_{2-d} - \dot{x}_2 = -k_2e_2 \\ \dot{e}_3 &= \dot{y}_{3-d} - \dot{x}_3 = -k_3e_3 \end{aligned}\right\}. \qquad (33)$$

Thence the control laws can be defined as:

$$\left.\begin{aligned} u_1 &= \dot{y}_{1-d} + k_1e_1 \\ u_2 &= \dot{y}_{2-d} + k_2e_2 \\ u_3 &= \dot{y}_{3-d} + k_3e_3 \end{aligned}\right\}. \qquad (34)$$

Furthermore, the coordinate transformation can be defined as:

$$\left.\begin{aligned} z_1 &= y_{1=}x_2 \\ z_2 &= y_2 = x_3 \\ z_3 &= y_{1=}x_1x_2 \end{aligned}\right\}. \qquad (35)$$

And the output lie derivative are:

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$$\left.\begin{aligned}\dot{y}_1 &= -x_1x_3 + x_2^2 + u_1\\ \dot{y}_2 &= x_1x_2 + u_2\\ \dot{y}_3 &= x_2^2 + x_1(-x_1x_3 + x_2^2 + u_1) + x_1u_3.\end{aligned}\right\} \tag{36}$$

Finally, the control law becomes:

$$u = G^{-1}(v - b), \tag{37}$$

Where, G^{-1} is the decoupling matrix inverse, *b* is the drift term and *v* is the new linear control input with tracking terms.

Simulation and Results

This section aims to apply two distinct methods input-output feedback linearization and input-state linearization to address the problem outlined in this paper. The goal is to enhance control and performance for the systems being studied. Furthermore, the results derived from both techniques will be compared to evaluate their effectiveness and applicability to the specific problem at hand. This comparative evaluation will offer critical insights into the advantages and limitations of each method, helping to identify the most suitable approach for tackling similar challenges in the future.

Task (1)

Simulation was conducted using the input-output feedback linearization technique. The results of the simulation for the described problem are illustrated in Figure 2 (a, b, c, and d).

The observed rms errors obtained in Figure 2(d); ranged from (0.062745 for e_1 , 0.100392 for e_2, and 0.012550 for e_3) this means the outputs tracking the reference inputs smoothly and the controls linearized the nonlinear system as desired, and the input-output linearization method works great for this system.
These low RMS values indicate that the outputs y_1, y_2, y_3 closely follow their respective reference trajectories y_{1-d}, y_{2-d}, y_{3-d} with minimal deviation. The small magnitude of e_3 (0.01255) suggests near-perfect tracking for the third output, likely due to the system's decoupled dynamics or reduced nonlinear coupling in that channel. Conversely, the slightly higher error for e_2 (0.10039) may reflect residual nonlinear interactions or unmodeled disturbances in the second output channel, which could be mitigated by refining the feedback gains or incorporating adaptive terms.
The success of the input-output linearization is further demonstrated by the smooth convergence of all outputs to their references,

confirming that the nonlinearities were effectively canceled by the control law.

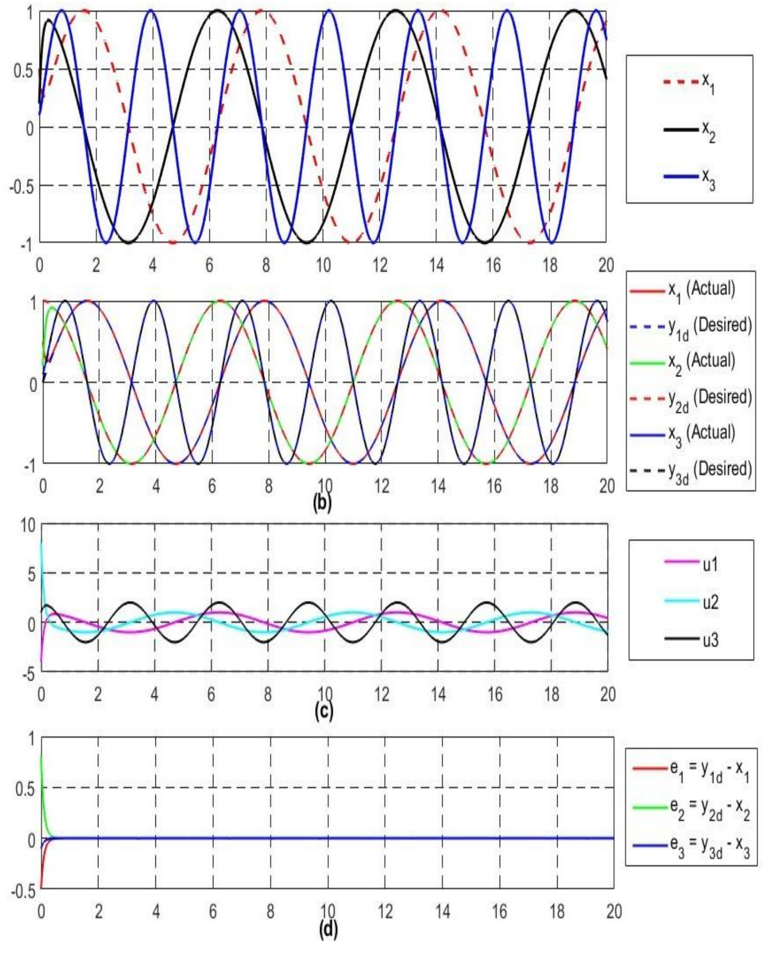

Figure 2. Simulation using input-output feedback linearization technique (a) State Trajectories (b) Output Tracking Performance (c) Control Inputs (d) Tracking Errors.

Task (2)

Simulation was performed using the input-state feedback linearization technique. For Task (2), the same parameters as those used in Task (1) were applied. The simulation results for the problem discussed in this paper are shown in Figure 3 (a, b, c, and d).

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The observed rms errors in Figure 3 ranged from (0.011715 for e_1 , 0.0195476 for e_2, and 0.009652 for e_3). This indicates that the outputs tracked the reference inputs smoothly, confirming that the controls effectively linearized the nonlinear system as intended. Therefore, the input-state linearization method performs well for this system.

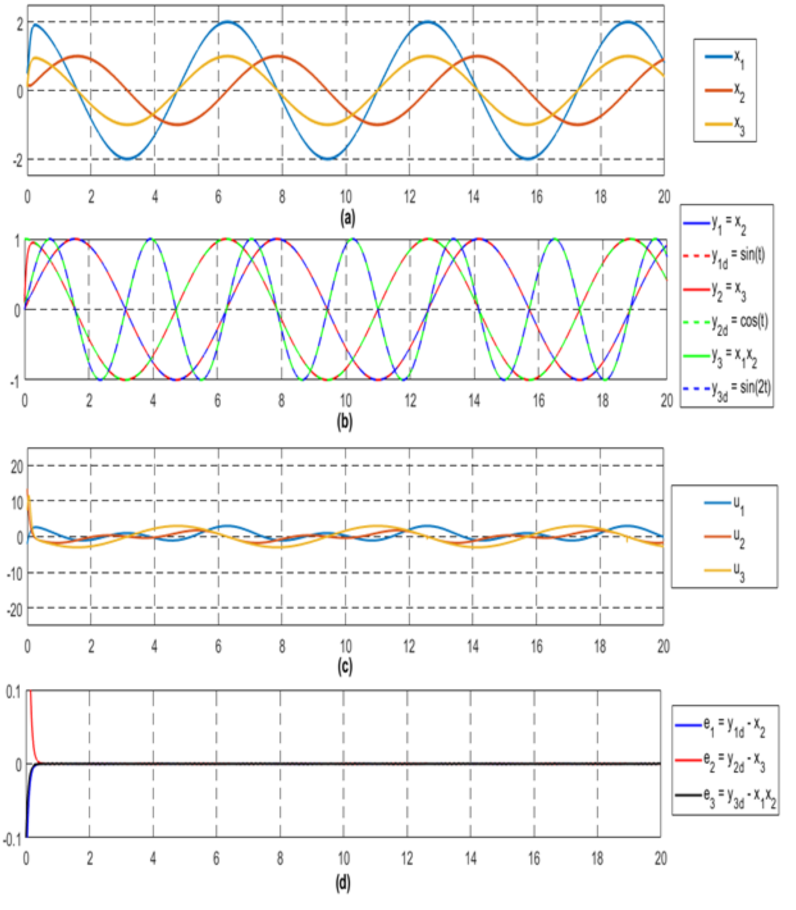

Fig. 3. Simulation using input-state feedback linearization technique (a) State Trajectories (b) Output Tracking Performance (c) Control Inputs (d) Tracking Errors.

Task (3)

In this task, we have compared the obtained results between two methods for this system:

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1.Key differences between the two methods

TABLE 1. Key differences between the two methods

Aspect	Input-Output Linearization (I/O)	Input-State Linearization (I/S)
Scope	Linearizes only the input-output map	Linearizes the entire state-space
Internal Dynamics	Leaves unobservable dynamics (zero dynamics) uncontrolled	Eliminates all nonlinearities in the state equations
Tracking Focus	Direct output tracking	Full state tracking (indirect output tracking)
Complexity	Requires fewer computations (output derivatives only)	Requires full-state transformation and Lie derivatives
Stability	Depends on zero dynamics stability	Guaranteed stability if linearization is exact

2. RMS Error Comparison

TABLE 2. RMS Error Comparison

Error	I/O Linearization	I/S Linearization	Comparison
e_1	0.062745	0.011715	I/S better
e_2	0.100392	0.0195476	I/S better
e_3	0.012550	0.009652	I/S better

3. Why Input-State Linearization Performs Better

a) Full State Control

- I/S linearization cancels all nonlinearities in the state equations, ensuring the entire system behaves linearly.
- I/O linearization only enforces a linear input-output relationship, leaving internal dynamics (e.g. x_1 in this system) unregulated. These unregulated states can destabilize outputs indirectly.

b) Error Dynamics

- The error dynamics for I/S linearization are explicitly defined as

$\dot{e} = -k_p\, e,$ guaranteeing exponential convergence to zero.

- In I/O linearization, error convergence depends on the stability of internal dynamics (e.g. x_1), which may introduce residual errors.

c) Control Authority

- I/S linearization uses full-state feedback to compute control inputs, allowing direct cancellation of cross-coupling terms (e.g. $x_1 x_3$ in $\dot{x}_2$).

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- I/O linearization relies on output derivatives, which may not fully decouple the system.

Conclusions and Future Work

In summary, input-state feedback linearization transforms the system dynamics by adjusting the internal state variables, whereas input-output feedback linearization achieves linearization by manipulating the input and output variables. The selection of the appropriate method depends on factors such as the accessibility and ease of measuring the system's internal states, as well as the specific control objectives. Both techniques yield closely aligned and satisfactory results for the problem examined in this study. However, based on the findings, input-state feedback linearization demonstrates greater accuracy compared to input-output feedback linearization. This is primarily due to the feasibility of direct measurement in the applications considered. In general, input-state feedback linearization proves to be more suitable than input-output feedback linearization in practical scenarios where direct measurement or reliable estimation of the system's internal states is achievable.

For future work, we intend to explore linearization approaches for nonlinear adaptive systems characterized by incomplete or uncertain parameters and dynamic knowledge. Such systems introduce distinct complexities that necessitate customized techniques for successful linearization and regulation. Investigating this intersection aims to deepen theoretical insights and formulate practical solutions for operating in environments with inherent uncertainties.

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